Recitation 14

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Review

Orthogonal diagonalization It is just like the usual diagonalization, but orthogonal. It can only be done to a **symmetric matrix** A. Then you do

- Find eigenvalues of A.
- Find eigenvectors.
- For each eigenvalue, if you have several lin. independent eigenvectors, orthogonalize them.
- Normalize all the eigenvectors you have. They would be columns of the matrix called P.
- **Note:** the order of eigenvectors in P should correspond to the order of eigenvalues, as before.
- Then $D = P^T AP$ is the orthogonal diagonalization of A. **Note:** matrix D is the same diagonalization we did a million times before. The main difference is that P is now an orthogonal matrix.

Singular value decomposition This is a decomposition $A = U\Sigma V^T$ (I have no idea why the notation is so weird) of an $m \times n$ matrix A (**not necessarily square**), where U is orthogonal $m \times m$ matrix, V is orthogonal $n \times n$ matrix, and Σ is a "diagonal matrix", i.e. a matrix of the form

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

with D being $r \times r$ honest diagonal matrix, having **non-increasing strictly positive** entries on the diagonal. The number r is the rank of A.

The **main idea** is again the change of basis. We can find a new nice **orthogonal** basis (columns of U) in $\mathbb{R}^m$, a new nice **orthogonal** basis (columns of V) in $\mathbb{R}^n$ such the matrix of A relative to the two bases is exactly Σ, a very simple looking matrix. This is the same idea as for a general change of basis, or for diagonalization, or for orthogonal diagonalization.

Indeed, since V is orthogonal, $V^T = V^{-1}$. If we call Σ by the name M, V by the name P, and U by the name Q, then the equation $A = U\Sigma V^T$ just says that $A = QMP^{-1}$, i.e. $M = Q^{-1}AP$, and that's exactly the change of basis formula which we know and love!

How to find SVD Suppose you want to find an SVD of a matrix A. These are the steps.

1. Find eigenvalues $\lambda_1, \dots, \lambda_n$ of $A^T A$, and **order them** to be $\lambda_1 \geq \dots \geq \lambda_n$.
2. Singular values are $\sigma_i = \sqrt{\lambda_i}$. These are the guys on the diagonal of Σ.
3. Find **orthonormal** eigenvectors $v_1, \dots, v_n$ of $A^T A$ corresponding to λ_i's. These vectors will be the **columns of the matrix** V. So by now you have both Σ and V. Need to find U.
4. To construct U, first r columns of U can be taken to be $u_1 := \frac{1}{\sigma_1} Av_1, \dots, u_r := \frac{1}{\sigma_r} Av_r$.
5. To construct the rest of the columns of U (if you don't have enough yet) just pick any vectors $u_{r+1}, \dots, u_m$ which have length 1 and so that $u_1, \dots, u_m$ are actually orthonormal.
6. Stare proudly at this one beautiful SVD you have just constructed.

Pseudoinverse If you know an SVD of a matrix A, $A = U\Sigma V^T$ and $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ with $r \times r$ matrix D having non-zero entries along the diagonal, then the **pseudoinverse** A^+ of A is given by $\mathbf{A^+ = U_r D^{-1} V_r^T}$, where U_r is the first r columns of $U = [u_1 \dots u_m]$ and V_r is the first r columns of $V = [v_1 \dots v_n]$.

Problems

Problem 1. Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ be two vectors in $\mathbb{R}^4$. Find two vectors v_3, v_4 such that $\{v_1, v_2, v_3, v_4\}$ is an orthogonal basis of $\mathbb{R}^4$.

Problem 2. Find an SVD of the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

Problem 3. Find an SVD of the matrix

$$A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Problem 4. Find the pseudoinverse of the matrix A from Problem 3.

Problem 5. Which of the following quadratic forms are positive-definite, negative-definite, or neither? Which are semi-definite?

1. $Q_1(x) = 3x_1^2 - 2x_1x_2 + x_2^2$ on $\mathbb{R}^2$;
2. $Q_2(x) = 3x_1^2 - 2x_1x_2 + x_2^2$ on $\mathbb{R}^3$;
3. $Q_3(x) = 6x_1x_2 + 4x_1x_3$ on $\mathbb{R}^3$;
4. $Q_4(x) = -x_1^2 + 2x_1x_2$ on $\mathbb{R}^2$.

Problem 6. Let $A = U\Sigma V^T$ be a SVD of an $m \times n$ matrix A.

Prove that if A is a square matrix, then $|\det A|$ is the product of singular values of A.

Prove that the columns of V are eigenvectors of A^TA, and the columns of U are eigenvectors of AA^T.

Problem 7. Prove that for any $m \times n$ matrix A defining $A \colon \mathbb{R}^n \to \mathbb{R}^m$, you can always find a basis of $\mathbb{R}^n$ and a basis of $\mathbb{R}^m$, relative to which the matrix A becomes

$$\Sigma' = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

where I is $r \times r$ **identity** matrix. In other words, you can always find $n \times n$ matrix P and $m \times m$ matrix Q such that $A = Q\Sigma' P^{-1}$.